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A NOTE OF REAL PARTS OF SOME SEMI-HYPONORMAL OPERATORS

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Let $\mathcal{H}$ be Hilbert space and $B(\mathcal{H})$ be set of all bounded linear operators on $\mathcal{H}$. Then for $T \in B(\mathcal{H})$

$$T\text{: semi-hyponormal} \iff |T| \geq |T^*|.$$

About semi-hyponormal operators, we have following 3 problems:

$$(1) \qquad \operatorname{Re} \sigma(T) = \sigma(\operatorname{Re} T) \quad ?$$

$$(2) \qquad \operatorname{conv}\sigma(T) = \overline{W(T)} \quad ?$$

$$(3) \quad \|(T-z)^{-1}\| \leq \frac{1}{\mathrm{d}(z, \sigma(T))} \quad \text{for every } z \notin \sigma(T) \quad ?$$

We have 2 kinds of concrete examples of semi-hyponormal operators.
D. Xia provides interesting examples (see [1],[5]). Let $\ell^2(\mathbf{Z})$ be the Hilbert space of all doubly-infinite sequences $a = \{a_k\}$ of complex numbers such that

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$||a||^2 = \sum_{k=-\infty}^{\infty} |a_k|^2 < \infty$, and let V be the bilateral shift: $(Va)_k = a_{k-1}$. Let $\mathcal{K}$ be a Hilbert space and let $\mathcal{H}$ be the Hilbert space of all doubly-infinite sequences $x = \{x_k\}$ of elements of $\mathcal{K}$ such that $||x||^2 = \sum_{k=-\infty}^{\infty} ||x_k||^2 < \infty$. Then we have $\mathcal{H} = \ell^2(\mathrm{Z}) \otimes \mathcal{K}$. Let $e_m = \{a_k\} \in \ell^2(\mathrm{Z})$ such that $a_m = 1$ and 0's elsewhere. Every $x = \{x_k\} \in \mathcal{H}$ has the representation $\sum_{k=-\infty}^{\infty} e_k \otimes x_k$. Let $\{A_k\}$ be a doubly-infinite sequence of positive operators on $\mathcal{K}$ such that $\{||A_k||\}$ is bounded. We define bounded operators A and U on $\mathcal{H}$ by

$$Ae_k \otimes x_k = e_k \otimes A_k x_k, \text{ and } Ue_k \otimes x_k = e_{k+1} \otimes x_k \ (k = 0, \pm 1, \pm 2, \cdots),$$

respectively. Then U has the form $V \otimes id_{\mathcal{K}}$. Put $T = UA$. Such an operator is called an operator valued bilateral weighted shift [3]. If positive operators $\{A_k\}$ satisfy that $A_{k+1} \geq A_k$ for every k and there exists j such that $A_{j+1}^2 \not\geq A_j^2$, then T is semi-hyponormal but not hyponormal.

Next second example is as follows: Let S on $\ell^2(\mathrm{Z})$ defined by $S = V(P + I + \frac{1}{2}(V + V^*))$, where P denotes the orthogonal projection from $\ell^2(\mathrm{Z})$ onto the subspace generated by $\{e_0, e_1, e_2, \cdots\}$. Xia showed that S is semi-hyponormal but not hyponormal [8, Chapter 3, Corollary 1.4].

2. Spectral properties.

Lemma 1. *Let T be an operator valued bilateral weighted shift. Then there exists a closed set F of positive real numbres such that*

$$\sigma(T) = \{z : |z| \in F\}.$$

Proof. Let $c \in \mathrm{C}$ such that $|c| = 1$. By [6, p. 52, Corollary 2] T and cT are unitarily equivalent. The proof follows from this property.

Theorem 2. *Let T be an operator valued bilateral weighted shift such that $r(T) = ||T||$. Then*

$$\text{conv } \sigma(T) = \overline{W(T)} \quad (i.e., T \textit{ is convexoid.})$$

and

$$\sigma(\text{Re } T) = \text{Re } (\sigma(T)).$$

Proof. Let $x \in \sigma(\text{Re } T)$. Suppose that $x \notin \text{Re } \sigma(T)$. Let L be the line $\text{Re } z = x$. Then L is disjoint from $\sigma(T)$. Suppose that $\sigma(T)$ is on the left side of L. There exists ε (> 0) such that $\text{Re } \sigma(T) \leq x - \varepsilon$. For any complex number $\lambda = |\lambda| e^{i\theta}$, we can choose $z \in \sigma(T)$ such that $z = \|T\| e^{i\theta}$ by Lemma 1. Since $(\|T\| + |\lambda|)e^{i\theta} \in \sigma(T + \lambda I)$, we have

$$r(T + \lambda I) \geq \|T\| + |\lambda| \quad (\geq \|T + \lambda I\|).$$

Hence we have $r(T + \lambda I) = \|T + \lambda I\|$, that is, T is a transaloid. Therefore by [3] or [5, Theorem 6.15.11] we have

$$\text{conv } \sigma(T) = \overline{W(T)}.$$

Thus

$$\text{conv } \sigma(\text{Re } T) = \overline{W(\text{Re } T)} = \text{Re } \overline{W(T)} = \text{Re conv } \sigma(T) \leq x - \varepsilon.$$

This implies that $x \leq x - \varepsilon$, which is a contradiction. We proceed similarly in case $\sigma(T)$ is on the right side. Therefore $\sigma(\text{Re } T) \subseteq \text{Re } \sigma(T)$.

Let $x \in \text{Re } \sigma(T)$. By Lemma 1, there exists $z \in \sigma(T)$ such that $\text{Re } z = x$ and $|z| = \|T\|$. Since z is a boundary point of $\sigma(T)$, there exists a sequence $\{f_n\}$ of unit vectors such that $\lim_{n\to\infty} \|(T - zI)f_n\| = 0$. By [5, Lemma 7.5.2], we have that $\lim_{n\to\infty} \|(T^* - \bar{z}I)f_n\| = 0$. Hence

$$\lim_{n\to\infty} \|(\text{Re } T - xI)f_n\| = 0,$$

so that $\text{Re } \sigma(T) \subseteq \sigma(\text{Re } T)$. Therefore, $\text{Re } \sigma(T) = \sigma(\text{Re } T)$.

In general, it holds that if T is semi-hyponormal, then $r(T) = \|T\|$. Hence we have

Corollary 3. *Let T be a semi-hyponormal operator valued bilateral weighted shift. Then*

$$\text{conv } \sigma(T) = \overline{W(T)} \text{ and } \sigma(\text{Re} T) = \text{Re}(\sigma(T)).$$

Theorem 4. *With the notations in the introduction, let* $S = V(P + I + \frac{1}{2}(V + V^*))$. *Then we have*

$$\sigma(\text{Re } S) = \text{Re } \sigma(S).$$

Proof. It F is a proper closed subset of $[0, 2\pi]$ such that $m([0, 2\pi]) = m(F)$. Since $[0, 2\pi] - F$ contains an open interval (a, b), we have $m([0, 2\pi]) - m(F) \geq m((a, b)) > 0$. This is a contradiction. Hence there exits no proper closed set F such that $m(F) = m([0, 2\pi])$. Applying [8, Chapter 4, Example 4.1] with $\alpha(\cdot) = 1$ and $\beta(\cdot) = 1 + \cos\theta$, we have that

$$\sigma(S) = \{e^{i\theta}(1 + \cos\theta + k) \ : \ 0 \leq k \leq 1, \ 0 \leq \theta \leq 2\pi\}.$$

Hence Re $\sigma(S) = \{(1 + k)\cos\theta + \cos^2\theta \ : \ 0 \leq k \leq 1, 0 \leq \theta \leq 2\pi \ \} = [-1, 3]$. Since S is semi-hyponormal, it holds that $\sigma_a(S) = \sigma_{na}(S)$. Hence we have

$$\text{Re } \sigma(S) \subseteq \sigma(\text{Re } S).$$

Next we will prove that $\sigma(\text{Re } S) \subseteq [-1, 3]$. First by the definition of S, we have $||\text{Re } S|| \leq ||S|| \leq 3$. Since Re S is convexoid, we may only prove $(\text{Re } S) + I \geq 0$.
Since Re S can be canonically represented by a matrix form with real components, for $\lambda \in \sigma(\text{Re } S)$ we choose a sequence $\{f_m\}$ of unit vectors in $\ell^2(\mathbf{Z})$ with real components such that $\lim_{m\to\infty} ||((\text{Re } S) - \lambda I)f_m|| = 0$. Since

$$2\text{Re } S = (V + V^*) + \frac{1}{2}(V^2 + V^{*2}) + (VP + PV^*) + VV^*,$$

we have, for $f = (\alpha_n)$ with all $\alpha_n \in \mathbf{R}$,

$$\begin{aligned}
2(((\text{Re } S) + I)f, f) &= ((V + V^*)f, f) + \frac{1}{2}((V^2 + V^{*2})f, f) \\
&\quad +((VP + PV^*)f, f) + (VV^*f, f) + 2(f, f) \\
&= 2\sum_{n=-\infty}^{\infty} \alpha_n\alpha_{n+1} + \sum_{n=-\infty}^{\infty} \alpha_n\alpha_{n+2} + 2\sum_{n=0}^{\infty} \alpha_n\alpha_{n+1} + 3\sum_{n=-\infty}^{\infty} \alpha_n^2 \\
&= \sum_{n=-\infty}^{\infty} \alpha_n\alpha_{n+2} + 4\sum_{n=-\infty}^{\infty} \alpha_n\alpha_{n+1}
\end{aligned}$$

$$-2\sum_{n=-\infty}^{-1}\alpha_n\alpha_{n+1}+3\sum_{n=-\infty}^{\infty}\alpha_n^2.$$

If we can choose a sequence $\{(a_n,b_n,c_n)\}_{n=-\infty}^{\infty}$ of triplets of positive numbers satisfying

$$2(((\mathrm{Re}\,S)+I)f,f)=\sum_{n=-\infty}^{\infty}(a_n\alpha_n+b_n\alpha_{n+1}+c_n\alpha_{n+2})^2$$

$$=\sum_{n=-\infty}^{\infty}(a_{n+2}^2+b_{n+1}^2+c_n^2)\alpha_n^2+\ 2\sum_{n=-\infty}^{\infty}(a_nb_n+b_{n-1}c_{n-1})\alpha_n\alpha_{n+1}$$

$$+\ 2\sum_{n=-\infty}^{\infty}(a_nc_n)\alpha_n\alpha_{n+2},$$

then we have $(\mathrm{Re}\,S)+I\geq 0$ and we hence can finish the proof.
For $n\geq -1$, since

$$(i)\ a_{n+2}^2+b_{n+1}^2+c_n^2=3,\quad (ii)\ 2(a_{n+1}b_{n+1}^2+b_nc_n)=4\ \text{ and }\ (iii)\ 2a_nc_n=1,$$

we define

$$a_n=\frac{1}{\sqrt{2}},\ b_n=\sqrt{2}\text{ and }c_n=\frac{1}{\sqrt{2}}.$$

For $n\leq -2$, since

$$(i)\ a_{n+2}^2+b_{n+1}^2+c_n^2=3,\ (ii)\ 2(a_{n+1}b_{n+1}+b_nc_n)=2\ \text{ and }\ (iii)\ 2a_nc_n=1,$$

inductively we define, in the following order:

$$c_n=\sqrt{3-a_{n+2}^2-b_{n+1}^2},\ b_n=\frac{1-a_{n+1}b_{n+1}}{c_n}\quad\text{and}\quad a_n=\frac{1}{2c_n}.$$

For a definition of c_n, we need to cheek that $3>a_{n+2}^2+b_{n+1}^2$. We caluculate

$$\begin{array}{lllllllll}
c_{-2} &=& \frac{1}{\sqrt{2}}, & b_{-2} &=& 0, & a_{-2} &=& \frac{1}{\sqrt{2}}\\
c_{-3} &=& \sqrt{\frac{5}{2}}, & b_{-3} &=& \sqrt{\frac{2}{5}}, & a_{-3} &=& \sqrt{\frac{1}{10}}\\
c_{-4} &=& \sqrt{\frac{21}{10}}, & b_{-4} &=& 4\sqrt{\frac{2}{105}}, & a_{-4} &=& \sqrt{\frac{5}{42}}\\
c_{-5} &=& \sqrt{\frac{109}{42}}, & b_{-5} &=& 17\sqrt{\frac{2}{2289}}, & a_{-5} &=& \sqrt{\frac{21}{218}}\\
c_{-6} &=& \sqrt{\frac{573}{218}}, & b_{-6} &=& 92\sqrt{\frac{2}{62457}}\text{ and} & a_{-6} &=& \sqrt{\frac{109}{1146}}
\end{array}$$

Then we have that

$$1.61 \leq c_{-5}, c_{-6} \leq 1.64,\ \ 0.50 \leq b_{-5},\ \ b_{-6} \leq 0.53,\ \ 0.30 \leq a_{-5}, a_{-6} \leq 0.32,$$

$$1.64 \leq \sqrt{3 - 0.53^2 - 0.32^2} \leq c_{-7} \leq \sqrt{3 - 0.50^2 - 0.30^2} \leq 1.64,$$

$$0.50 \leq \frac{1 - 0.53 \times 0.32}{1.64} \leq b_{-7} \leq \frac{1 - 0.50 \times 0.30}{1.61} \leq 0.53$$

and

$$0.30 \leq \frac{1}{2 \cdot 1.64} \leq a_{-7} \leq \frac{1}{2 \cdot 1.61} \leq 0.32.$$

Thus we can define c_n, b_n and a_n for $n \leq -8$. This completes the proof.

By a similar argument in Theorem 2, we have that Im $\sigma(T) = \sigma(\text{Im } T)$ for T of Theorem 2. In the proof of Theorem 4 we regareded Re S as an infinite matrix with real components.

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